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THREE-DIMENSIONAL CONTACT PROBLEM FOR A PRESTRESSED ELASTIC BODY

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A half-space of an incompressible neo —Hookean [1, 2] material subjected to a homogeneous bi-axial tension or compression along its boundary, is considered. A small deformation caused by the action of a smooth rigid stamp on the boundary of the half-space is superimposed on the initial finite deformation. An integral equation is obtained for the contact pressure. A solution of this equation is obtained for an inclined elliptic stamp with a flat base, and for an elliptic stamp with a curved base, for the cases when the extension coefficients in two directions are either identical, or differ little from each other. The influence of the inital loading on the distribution of the contact pressure, the displacement of the stamp and the form of the contact zone, is analysed.

1. Using the relations of the theory of small deformations of an elastic body superimposed on a finite deformation [2], we obtain the following equations describing the deformation of a prestressed, neo —Hookean body:

$$G\left(\lambda_1^2 \frac{\partial^2 \mathbf{u}}{\partial x^2} + \lambda_2^2 \frac{\partial^2 \mathbf{u}}{\partial y^2} + \lambda_1^{-2}\lambda_2^{-2} \frac{\partial^2 \mathbf{u}}{\partial z^2}\right) + 2\,\mathrm{grad}\, p = 0 \tag{1.1}$$

$$\operatorname{div} \mathbf{u} = 0, \quad \mathbf{u} = (u, v, w)$$

Here x, y, z are the Cartesian coordinates in the initial deformed state, $\mathbf{u}$ is the displacement vector, p is the pressure function appearing in (1.1) by virtue of the incompressibility of the material, λ_1 and λ_2 are the coefficients of the preliminary extension in the x- and y-directions, and G is a material constant equal to the shear modulus when the deformations are small.

Equations (1.1) have been obtained under the assumption that in the initial state of stress $\sigma_z = 0$. The remaining two stresses are given in terms of the extension coefficients, by the formulas

$$\sigma_x = G(\lambda_1^2 - \lambda_1^{-2}\lambda_2^{-2}), \quad \sigma_y = G(\lambda_2^2 - \lambda_1^{-2}\lambda_2^{-2})$$

The boundary conditions at the boundary of the half-space $z = 0$ are, in the case of penetration of a smooth rigid stamp are as follows:

$$\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0, \quad \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0, \quad p + G\lambda_1^{-2}\lambda_2^{-2} \frac{\partial w}{\partial z} = 0 \tag{1.2}$$

in the area outside the region of contact, and

$$\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}=0, \quad \frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}=0, \quad w=\delta-\theta_2 x+\theta_1 y-\varphi(x, y) \quad (1.3)$$

within the region of contact S. Here δ denotes the displacement of the stamp, θ_1 and θ_2 are the cosines of the angles of inclination of the stamp [2] and $\varphi(x, y)$ is a quadratic function describing the surface of the stamp base.

First we shall consider the problem of a concentrated, unit normal force applied at the coordinate origin and acting on the boundary of the half-space. Applying to (1.1) the two-dimensional Fourier transform

$$\bar{f}(\alpha, \beta, z)=\iint_{-\infty}^{\infty} f(x, y, z) \exp[-i(\alpha x+\beta y)]\, dx\, dy$$

we obtain the transform of the normal displacement at the points of the boundary

$$\bar{w}(\alpha, \beta, 0)=\frac{\mu \lambda_1^2 \lambda_2^2(\mu^2-\lambda_1^2 \lambda_2^2 \nu^2)}{G[\mu^2-\lambda_1^4 \lambda_2^4 \nu^4-2 \mu^2(\mu-\lambda_1 \lambda_2 \nu)^2]} \equiv W(\alpha, \beta) \quad (1.4)$$

$$\mu=\sqrt{\alpha^2+\beta^2}, \quad \nu=\sqrt{\lambda_1^2 \alpha^2+\lambda_2^2 \beta^2}$$

The integral equation for the constact pressure $q(x, y)$ follows from (1.3) and (1.4)

$$\delta-\theta_2 x+\theta_1 y-\varphi(x, y)= \quad (1.5)$$

$$\frac{1}{(2\pi)^2} \iint_S q(\xi, \eta) \iint_{-\infty}^{\infty} W(\alpha, \beta) \exp\{i[\alpha(x-\xi)+$$

$$\beta(y-\eta)]\}\, d\alpha\, d\beta\, d\xi\, d\eta$$

2. First we consider the case when the preliminary extension or compression is the same in both directions ($\lambda_1=\lambda_2=\lambda$). From (1.4) we obtain

$$\bar{w}(\alpha, \beta, 0)=\frac{N(\lambda)}{2G\sqrt{\alpha^2+\beta^2}}, \quad w(x, y, 0)=\frac{N(\lambda)}{4\pi G\sqrt{x^2+y^2}} \quad (2.1)$$

$$N(\lambda)=\frac{2\lambda^4(1+\lambda^3)}{\lambda^9+\lambda^6+3\lambda^3-1}$$

The integral equation (1.5) now assumes the form

$$\delta-\theta_2 x+\theta_1 y-\varphi(x, y)=\frac{1}{4\pi G} \iint_S \frac{N(\lambda)\, q(\xi, \eta)\, d\xi\, d\eta}{\sqrt{(x-\xi)^2+(y-\eta)^2}} \quad (2.2)$$

The kernel of (2.2) differs from the kernel of the classical contact problem [2–4] only in the value of the constant multiplier $N(\lambda)$. Since $N(1)=1$, in the absence of preliminary deformation the equation (2.2) coincides with the classical one (where, since the material is incompressible, the Poisson's ratio should be made equal to $1/2$).

Using (2.1) we can establish that the coefficient $N(\lambda)$ decreases monotonously over the interval $\lambda^* < \lambda < \infty$ where $\lambda^* \approx 0.667$. When $\lambda \to \lambda^*$, $N(\lambda)$ increases without bounds, i.e. when $\lambda \to \lambda^*$, the compressed half-space loses its stability. From this it follows that the values of the compression coefficient cannot be less than λ^* .

The solution of the integral equation (2.2) for the case of an elliptic stamp with a flat inclined base ($\varphi(x, y) = 0$) [2] shows that in the case of a straight (not inclined) stamp $(\theta_1 = \theta_2 = 0)$ a preliminary uniform extension does not affect the distribution of the contact pressure, but influences the distribution of the displacement of the stamp. For an inclined stamp the magnitude of the initial extention affects also the character of the contact pressure distribution. Similar phenomena were discovered in the plane contact problem [5].

In the case of an elliptic stamp with a curved base [2] we have

$$\theta_1 = \theta_2 = 0, \quad \varphi(x, y) = x^2 / (2R_1) + y^2 / (2R_2)$$

where we assume that $R_1 \geqslant R_2$. Just as in the case of an elliptic stamp with a flat inclined base, the solution of (2.2) is obtained from the solution corresponding to the unstressed half-space [2]. From the solution it follows that the initial deformation does not affect the form of the region of contact, nor the character of the distribution of the contact pressure. The size of the region of contact and the stamp displacement however, depend on the previous loading, and the preliminary extension (compression) reduces (increases) the size of the region of contact and the magnitude of the stamp displacement.

3. Le us assume that the extension coefficients in two directions differ little from each other, and let us set

$$\lambda = {}^1/_2\,(\lambda_1 + \lambda_2), \quad \chi = {}^1/_2\,(\lambda_2 - \lambda_1) \tag{3.1}$$

Assuming that $|\chi| \ll 1$, we simplify the expression (1.4) retaining only the first order terms in χ . We obtain

$$\overline{w}(\alpha, \beta, 0) = \frac{1}{2G}\left[\frac{N(\lambda)}{(\alpha^2+\beta^2)^{1/2}} + \chi L(\lambda)\frac{\alpha^2-\beta^2}{(\alpha^2+\beta^2)^{3/2}}\right] + O(\chi^2) \tag{3.2}$$

$$L(\lambda) = \frac{4\lambda^6(\lambda^9 + 2\lambda^6 + \lambda^3 + 2)}{\lambda^9 + \lambda^6 + 3\lambda^3 - 1}$$

The coefficient $L(\lambda)$ is positive within the interval $\lambda^* < \lambda < \infty$ in question.

Using the known [6] formula for the Fourier transform of a function of the form $(x^2 + y^2)^{n/2}$ we obtain from (3.2) $w(x, y, 0)$. The integral equation for the contact pressure assumes the following form with the accuracy of up to the terms of order χ^2 :

$$\delta - \theta_2 x + \theta_1 y - \varphi(x, y) = \frac{1}{4\pi G}\iint_S q(\xi, \eta)\left[\frac{N(\lambda)}{[(x-\xi)^2 + (y-\eta)^2]^{1/2}} + \right. \tag{3.3}$$

$$\left. \chi L(\lambda)\frac{(y-\eta)^2 - (x-\xi)^2}{[(x-\xi)^2 + (y-\eta)^2]^{3/2}}\right] d\xi\, d\eta$$

Using the examples described in [2 — 4, 7], we can obtain the formulas on which the solution of (3.3) is based, e.g.

$$\iint_S \frac{(y-\eta)^2-(x-\xi)^2}{r^3 R}\,\xi d\xi d\eta = \frac{2\pi a\sqrt{1-e^2}}{e^4}\,x\,[(4-e^2)\,\mathbf{E}-(4-3e^2)\,\mathbf{K}]x$$

$$\iint_S \frac{(y-\eta)^2-(x-\xi)^2}{r^3 R}\,\eta d\xi d\eta = \frac{2\pi a\sqrt{1-e^2}}{e^4}\,y\,[(1-e^2)(4-e^2)\,\mathbf{K}-(4-3e^2)\,\mathbf{E}]$$

$$\iint_S \frac{R}{r^3}\,[(y-\eta)^2-(x-\xi)^2]\,d\xi d\eta = \frac{\pi a\sqrt{1-e^2}}{e^2}\Big\{[2\mathbf{E}-(2-e^2)\times \mathbf{K}]-\frac{x^2}{a^2e^2}\,[(4-e^2)\,\mathbf{E}-(4-3e^2)\,\mathbf{K}]-\frac{y^2}{a^2e^2(1-e^2)}\times [(1-e^2)(4-e^2)\,\mathbf{K}-(4-3e^2)\,\mathbf{E}]\Big\}$$

$$r=\{(x-\xi)^2+(y-\eta)^2\}^{1/2},\qquad R=\{1-\xi^2/a^2-\eta^2/[a^2(1-e^2)]\}^{1/2}$$

The integration extends over the elliptic area stretched along the x-axis, with the major semiaxis a and excentrcity e. The argument e has been omitted from the expressions for the complete elliptic integrals $\mathbf{E}$ and $\mathbf{K}$ for brevity.

We seek a solution of (3.3) for a flat elliptic stamp with an inclined base in the form

$$q(\xi,\eta)=(a_0+a_1\xi+a_2\eta)\,\{1-\xi^2/a^2-\eta^2/[a^2(1-e^2)]\}^{-1/2} \qquad (3.4)$$

Using the conditions of balance between the stamp and the external force Q we obtain, to within the terms of order of χ^2:

$$a_0=\frac{Q}{2\pi a^2\sqrt{1-e^2}},\qquad \delta=\frac{QN(\lambda)\,\mathbf{K}}{4\pi Ga}(1+\chi f_0) \qquad (3.5)$$

$$a_1=-\frac{2Ge^2\theta_2}{N(\lambda)\,a\sqrt{1-e^2}\,(\mathbf{K}-\mathbf{E})}(1-\chi f_1)$$

$$a_2=\frac{2Ge^2\theta_1}{N(\lambda)\,a\sqrt{1-e^2}\,[\mathbf{E}-(1-e^2)\,\mathbf{K}]}(1-\chi f_2)$$

Here and henceforth we have

$$f_0=\frac{L(\lambda)}{N(\lambda)}\,\frac{2\mathbf{E}-(2-e^2)\,\mathbf{K}}{e^2\mathbf{K}},\qquad f_1=\frac{L(\lambda)}{N(\lambda)}\,\frac{(4-e^2)\,\mathbf{E}-(4-3e^2)\,\mathbf{K}}{e^2(\mathbf{K}-\mathbf{E})}$$

$$f_2=\frac{L(\lambda)}{N(\lambda)}\,\frac{(1-e^2)(4-e^2)\,\mathbf{K}-(4-3e^2)\,\mathbf{E}}{e^2\mathbf{E}-e^2(1-e^2)\,\mathbf{K}}$$

For a curved stamp of elliptic cross section in which the principal axes of the surface coincide with and x, y -axes, the solution of (3.3) satisfying the condition of equilibrium of the stamp, has the form

$$q(\xi,\eta)=\frac{3Q}{2\pi a^2\sqrt{1-e^2}}\left[1-\frac{\xi^2}{a^2}-\frac{\eta^2}{a^2(1-e^2)}\right]^{1/2} \qquad (3.6)$$

Substituting (3.6) into (3.3), we arrive at a system of equations for determining the constants e, a and δ

$$\delta = \frac{3QN(\lambda)\mathbf{K}}{8\pi Ga}(1+\chi f_0), \frac{1}{2R_1} = \frac{3QN(\lambda)(\mathbf{K}-\mathbf{E})}{8\pi Ga^3e^2}(1+\chi f_1) \qquad (3.7)$$
$$\frac{1}{2R_2} = \frac{3QN(\lambda)[\mathbf{E}-(1-e^2)\mathbf{K}]}{8\pi Ga^3e^2(1-e^2)}(1+\chi f_2)$$

and the solution of (3.7) should be sought in the form

$$e^2 = e_{(0)}{}^2 + \chi e_{(1)}{}^2 + \ldots, \qquad a = a_{(0)} + \chi a_{(1)} + \ldots \qquad (3.8)$$
$$\delta = \delta_{(0)} + \chi\delta_{(1)} + \ldots$$

where zero index indicates a known solution of the problem studied in Sect. 2 for a body subjected to uniform extension. Since the integral equation (3.3) holds with the accuracy only to within the terms of order χ^2, it is sufficient to compute in (3.8) $e_{(1)}$, $a_{(1)}$ and $\delta_{(1)}$.

As an example we shall compute the excentricity of the region of contact for a stamp of circular cross section $(R_1 = R_2)$. Dividing the second equation of (3.7) by the third, substituting into the resulting expression the expansion (3.8) and remembering that $e_{(0)} = 0$ we obtain, neglecting the terms of order of χ^2 and higher,

$$e^2 = \frac{4}{3}\chi\frac{L(\lambda)}{N(\lambda)} \qquad (3.9)$$

The right hand side of (3.9) is positive within the admissible interval $\lambda^* < \lambda < \infty$ when χ are positive. This means that the assumption made above that the region of contact S is stretched along the x-axis, is valid for $\lambda_2 > \lambda_1$. Thus the region of contact in the case of imbedding of a curved stamp of circular cross section, is bounded by an ellipse stretched in the direction of the axis which undergoes a smaller extension.

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